Question 1

1. *Let* $f(x) = e^x - 1$ *and let* f^{-1} *denote the inverse function. Then* $(f^{-1})'(e^2 - 1) =$

- We have the formula $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$. We apply this formula with $a = e^2 - 1$.
- Since $f(2) = e^2 - 1$, we have $f^{-1}(e^2 - 1) = 2$.
- $f'(x) = e^x$, therefore $f'(2) = e^2$.
- The formula says that $(f^{-1})'(e^2 - 1) = \frac{1}{f'(f^{-1}(e^2-1))} = \frac{1}{f'(2)} = \frac{1}{e^2} = e^{-2}$
-

Question 2

2. Solve the following equation for x:

$$\ln(x+4) - \ln x = 1 \ .$$

- Amalgamating the logarithms, our equation becomes:

$$\ln\left(\frac{x+4}{x}\right) = 1.$$

- Applying the exponential to both sides, we get

$$\left(\frac{x+4}{x}\right) = e^1 = e$$

- Multiplying both sides by x, we get $x + 4 = ex$ and $x - ex = -4$.
- Therefore $x(1-e) = -4$ and

$$x = \frac{-4}{1-e} = \frac{4}{e-1}.$$

Question 3

3. *Find the derivative of* $(x^2+1)^{x^2+1}$.

- We use logarithmic differentiation. Let $y = (x^2+1)^{x^2+1}$. Then

$$\ln y = (x^2+1)\ln(x^2+1).$$

- Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x^2+1)\ln(x^2+1) = 2x\ln(x^2+1) + \frac{2x(x^2+1)}{(x^2+1)} = 2x\Big[\ln(x^2+1)+1\Big].$$

- Multiplying both sides by y, we get

$$\frac{dy}{dx} = y2x\Big[\ln(x^2+1)+1\Big] = (x^2+1)^{x^2+1}2x\Big[\ln(x^2+1)+1\Big]$$

Question 4

4. $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}} =$
   - This is an indeterminate form of type 1^∞.
   - We have

$$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0^+} e^{\frac{\ln(\cos x)}{x^2}} = e^{\lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2}}$$

- $= \quad (by l'Hop) \quad e^{\lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x}} = e^{\lim_{x\to 0^+} \frac{-\tan x}{2x}}$
- $= \quad (by l'Hop) \quad e^{\lim_{x\to 0^+} \frac{-\sec^2 x}{2}} = e^{-1/2}$

Question 5

5. *The integral*

$$\int_0^{\pi/2} x\cos(x)dx$$

is

- We use integration by parts with $u = x$, $dv = \cos x dx$. We get $du = dx$ and $v = \sin x$.
- Recall that $\int udv = uv - \int vdu$. Therefore $\int_0^{\pi/2} x\cos xdx = x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin xdx$
- $= \frac{\pi}{2}\sin\frac{\pi}{2} - 0 - \Big[-\cos x\Big]_0^{\pi/2} = \frac{\pi}{2} + \Big[\cos\frac{\pi}{2} - \cos 0\Big]$
- $= \frac{\pi}{2} + \Big[0 - 1\Big] = \frac{\pi}{2} - 1.$

Question 6

6. *Evaluate*

$$\int \frac{x^2}{\sqrt{9-x^2}}\, dx.$$

- Here we use the trigonometric substitution $x = 3\sin\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- We have $x^2 = 9\sin^2\theta, \quad dx = 3\cos\theta d\theta$ and $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3|\cos\theta| = 3\cos\theta$
- $\int \frac{x^2}{\sqrt{9-x^2}} dx = \int \frac{9\sin^2\theta}{3\cos\theta} 3\cos\theta d\theta = 9\int \sin^2\theta d\theta.$
- $= \frac{9}{2}\int(1-\cos(2\theta))d\theta = \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right] + C$
- We have $\theta = \sin^{-1}\frac{x}{3}$. Therefore $\int \frac{x^2}{\sqrt{9-x^2}} dx = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{2\sin\theta\cos\theta}{2}\right] + C$
- Using a triangle, we get $\cos\theta = \frac{\sqrt{9-x^2}}{3}$ and $\int \frac{x^2}{\sqrt{9-x^2}} dx = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{\frac{2}{9}\, x\sqrt{9-x^2}}{2}\right] + C = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{9}\right] + C$

Question 7

7. *If you expand* $\frac{2x+1}{x^3+x}$ *as a partial fraction, which expression below would you get?*

a. $\frac{1}{x} + \frac{-x+2}{x^2+1}$

b. $\frac{2}{x} + \frac{1}{x^2+1}$

c. $\frac{-1}{x} + \frac{x}{x^2+1}$

d. $\frac{-1}{x^2} + \frac{1}{x+1}$

e. $\frac{-2}{x} + \frac{1}{x^2+1}$

- $\frac{2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$
- Multiplying the above equation by $x(x^2+1)$, we get $2x+1 = A(x^2+1)+x(Bx+C) = Ax^2+A+Bx^2+Cx = (A+B)x^2+Cx+A.$
- Comparing coefficients, we get $A = 1$, $C = 2$, and $A + B = 0$. Therefore $B = -A = -1$.
- The partial fractions decomposition of $\frac{2x+1}{x(x^2+1)}$ is therefore $\frac{1}{x} + \frac{-x+2}{x^2+1}$.

Question 8

8. The integral

$$\int_0^2 \frac{1}{1-x}dx$$

is

a. divergent b. 0 c. $\ln 2$
d. $\frac{\pi}{\sqrt{2}}$ e. $\frac{\pi}{6}$

- This is an improper integral $\int_0^2 \frac{1}{1-x}dx = \int_0^1 \frac{1}{1-x}dx + \int_1^2 \frac{1}{1-x}dx$
- $= \lim_{t\to 1^-} \int_0^t \frac{1}{1-x}dx + \lim_{t\to 1^+} \int_t^2 \frac{1}{1-x}dx.$
- If one of these integral diverges the original integral diverges.
- We have $\lim_{t\to 1^+} \int_t^2 \frac{1}{1-x}dx = \lim_{t\to 1^+}[-\ln|1-x|\Big|_t^2$
  $= \lim_{t\to 1^+}[-\ln|-1| + \ln|1-t|] = -\infty$
- Therefore the integral $\int_0^2 \frac{1}{1-x}dx$ diverges.

Question 9

9. *If 100 grams of radioactive material with a half–life of two days are present at day zero, how many grams are left at day three?*

- We have initial amount $m_0 = 100$ and half life $t_{\frac{1}{2}} = 2$ days.
- The amount left after t days is given by $m(t) = m_0 e^{kt} = 100e^{kt}$ for some constant k.
- To find the value of k, we use the fact that the half-life is 2 days. This tells us that $50 = 100e^{2k}$ or $\frac{1}{2} = e^{2k}$. Applying the natural logarithm to both sides, we get $\ln\frac{1}{2} = \ln e^{2k}$ or $-\ln 2 = 2k$.
- Therefore $k = \frac{-\ln 2}{2}$ and $m(t) = 100e^{-\frac{t\ln 2}{2}} = 100(e^{\ln 2})^{-\frac{t}{2}} = 100(2)^{-\frac{t}{2}}$
- After 3 days, we have $m(3) = 100(2)^{-\frac{3}{2}} = \frac{100}{3\sqrt{3}}$.

Question 10

10. *If* $x\frac{dy}{dx} + 3y = \frac{4}{x}$, *and* $y(1) = 10$, *find* $y(2)$.
    - We put the equation in standard form by dividing across by x. $\frac{dy}{dx} + \frac{3}{x}y = \frac{4}{x^2}$.
    - This is a first order linear differential equation.
    - The integrating factor is $e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$.
    - Multiplying the standard equation by x^3, we get $x^3\frac{dy}{dx} + 3x^2y = 4x$ or $\frac{d(x^3y)}{dx} = 4x$.
    - Integrating both sides with respect to x, we get $x^3y = 4\frac{x^2}{2} + C = 2x^2 + C$.
    - Dividing across by x^3, we get $y = \frac{2}{x} + \frac{C}{x^3}$
    - Using the initial value condition $y(1) = 10$, we get $10 = y(1) = 2 + C$ or $C = 8$.
    - Therefore $y = \frac{2}{x} + \frac{8}{x^3}$ and $y(2) = 1 + 1 = 2$.

Question 11

11. *The solution to the initial value problem*

$$y' = x\cos^2 y \qquad\qquad y(2) = 0$$

satisfies the implicit equation

a) $\tan(y) = \frac{x^2}{2} - 2$ *b*) $\frac{ey}{2} = e^{\cos x} - e^{\cos 2}$ *c*) $\cos y = x - 1$
d) $\cos(y) = x + \cos(2)$ *e*) $e^{2y+1} = \arcsin(x - 2) + e$

- This is a separable differential equation $\frac{dy}{dx} = x\cos^2 y$.
- We separate the variables $\frac{dy}{\cos^2 y} = xdx$
- We have $\int \sec^2 ydy = \int xdx$
- Therefore $\tan y = \frac{x^2}{2} + C$.
- Using the initial value condition, we get $y(2) = 0$ or $\tan 0 = \frac{2^2}{2} + C$, giving that $0 = 2 + C$ and $C = -2$.
- Therefore $\tan y = \frac{x^2}{2} - 2$.

Question 12

12. Use Euler's method with step size 0.1 to estimate $y(1.2)$ where $y(x)$ is the solution to the initial value problem

$$y' = xy + 1 \qquad y(1) = 0.$$

- $x_0 = 1, \quad y_0 = 0$
- $x_1 = x_0 + h = 1.1, \quad y_1 = y_0 + h(x_0 y_0 + 1) = 0 + (0.1)(1 \cdot 0 + 1) = 0.1$
- $x_2 = x_1 + h = 1.2, \quad y_2 = y_1 + h(x_1 y_1 + 1) = 0.1 + (0.1)((1.1)(0.1) + 1)$
- $= 0.1 + 0.1(0.11 + 1) = 0.1 + 0.1(1.11) = 0.1 + 0.111 = 0.211$

Question 13

13. Find $\sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}}$

a) $\frac{20}{3}$ *b*) $\frac{4}{15}$ *c*) $\frac{5}{4}$ *d*) $\frac{5}{3}$ *e*) $\frac{5}{12}$

- $\sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4^n}{3 \cdot 5^{n-1}} = \frac{4}{3} + \frac{4^2}{3 \cdot 5} + \dots$
- This is a geometric series with $a =$ 1st term $= 4/3$ and $r =$ (2nd term)/(1st term) $= 4/5$.
- Since $|r| < 1$, we have $\sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}} = \frac{a}{1-r} = \frac{4/3}{1-4/5} = \frac{4/3}{1/5} = \frac{20}{3}$.

Question 14

14. Which of the following series converge conditionally?

$$\text{(I)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} \qquad \text{(II)} \sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln n} \qquad \text{(III)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \text{ ?}$$

(III) converges conditionally, (I) and (II) do not converge conditionally
(I) and (II) converge conditionally, (III) does not converge conditionally
(I) and (III) converge conditionally, (II) does not converge conditionally
(II) and (III) converge conditionally, (I) does not converge conditionally
(II) converges conditionally, (I) and (III) do not converge conditionally

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
- $\sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln n}$ diverges by the divergence test, since $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} = (l'Hop) \lim_{x\to\infty} \frac{1}{1/x} = \infty$.
- $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test, however it does not converge absolutely since $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

Question 15

15. Which series below absolutely converges?

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+1)}$ c) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^3}$

d) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{n^2+1}$ e) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\pi^n}{3^n}$

- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.
- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+1)}$ does not converge absolutely since $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, ($n > \ln(n+1)$ for $n > 1$.)
- $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^3}$ does not converge absolutely since $\sum_{n=1}^{\infty} \frac{n!}{n^3}$ diverges by the ratio test.
  $(\lim_{n\to\infty} \frac{(n+1)!/(n+1)^3}{n!/n^3} = \lim_{n\to\infty}(n+1) \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 = \infty > 1.\ )$
- $\sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{n^2+1}$ does not converge by comparison with $\sum_{n=1}^{\infty} \frac{n^{3/2}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (which diverges because it is a p-series with $p < 1$).
- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\pi^n}{3^n}$ diverges since it is a geometric series with $|r| = \frac{\pi}{3} > 1$.

Question 16

16. The interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{\sqrt{n}}$$

is

a) $[-4,-2)$ b) $(-4,-2)$ c) $(-1,1)$ d) $(2,4)$ e) $[2,4]$

- Using the ratio test, we get $\lim_{n\to\infty} \frac{|x+3|^{n+1}/\sqrt{n+1}}{|x+3|^n/\sqrt{n}} = \lim_{n\to\infty} |x+3|\sqrt{\frac{n}{n+1}} = |x+3|$
- The ratio test says that the power series converges if $|x+3| < 1$ and diverges if $|x+3| > 1$. (R.O.C. $= 1$)
- The power series converges if $-1 < x+3 < 1$ or $-4 < x < -2$.
- We need to check the end points of this interval.
- When $x = -4$, we get $\sum_{n=1}^{\infty} \frac{(x+3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-4+3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which converges by the alternating series test.
- When $x = -2$, we get $\sum_{n=1}^{\infty} \frac{(x+3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-2+3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges since it is a p-series with $p = 1/2 < 1$.

Question 17

17. If $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(2n+1)!}$, find the power series centered at 2 for the function $\int_2^x f(t)\, dt$.

a) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^{n+1}}{(n+1)(2n+1)!}$ b) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^{n+1}}{(n^2)(2n+1)!}$ c) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^{2n+1}}{(n+1)(2n)!}$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^{n+1}}{(n+1)!}$

e) The given function can not be represented by a power series centered at 2.

- $\int_2^x f(t)\, dt$ is the unique antiderivative $F(x) = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(2n+1)!} dx$ with $F(2) = 0$.
- We have $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int (x-2)^n dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{(x-2)^{n+1}}{n+1} dx + C$.
- The condition that $F(2) = 0$ gives that $0 = F(2) = 0 + C$. Hence $C = 0$.
- Therefore $\int_2^x f(t) dt = F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{(x-2)^{n+1}}{n+1} dx$.

Question 18

18. Which series below is the MacLaurin series (Taylor series centered at 0) for $\frac{x^2}{1+x}$?

a) $\sum_{n=0}^{\infty}(-1)^n\ x^{n+2}$ b) $\sum_{n=0}^{\infty} x^{2n+2}$ c) $\sum_{n=0}^{\infty} \frac{x^{n+2}}{n+2}$
$\sum_{n=2}^{\infty} \frac{(-1)^n x^{2n-2}}{n!}$ e) $\sum_{n=0}^{\infty}(-1)^n x^{2n}$

- We have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.
- Using substitution we get $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$
- Multiplying by x^2, we get $\frac{x^2}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^{n+2}$.

Question 19

19. The following is the fourth order Taylor polynomial of the function $f(x)$ at a.

$$T_4(x) = 10 + 5(x-a) + \sqrt{3}(x-a)^2 + \frac{1}{2\pi}(x-a)^3 + 17e(x-a)^4$$

What is $f'''(a)$?

Solution: By the Taylor formula, we have $\frac{f'''(a)}{3!} = \frac{1}{2\pi}$ (which is the coefficient of $(x-a)^3$) and hence $f'''(a) = \frac{1 \cdot 2 \cdot 3}{2\pi} = \frac{3}{\pi}$.

Question 20

20. $\lim_{x\to 0} \frac{\sin(x^3)-x^3}{x^9} =$

Hint: Without MacLaurin series this may be a long problem.

a) $-\frac{1}{6}$ b) ∞ c) 0 d) $\frac{9}{7}$ e) $\frac{7}{9}$

- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$.
- Therefore $\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \dots$.
- Hence $\frac{\sin(x^3)-x^3}{x^9} = \frac{-\frac{x^9}{3!}+\frac{x^{15}}{5!}-\frac{x^{21}}{7!}+\dots}{x^9} = -\frac{1}{6} + \frac{x^6}{5!} - \frac{x^{12}}{7!} + \dots$.
- Therefore $\lim_{x\to 0} \frac{\sin(x^3)-x^3}{x^9} = \lim_{x\to 0}[-\frac{1}{6} + \frac{x^6}{5!} - \frac{x^{12}}{7!} + \dots] = -\frac{1}{6}$.

Question 21

21. Which line below is the tangent line to the parameterized curve

$$x = \cos t + 2\cos(2t), \qquad y = \sin t + 2\sin(2t)$$

when $t = \pi/2$?
a) $y = 4x + 9$ b) $y = -4x - 7$
c) $y = x + 3$ d) $y = -x + 3$ e) $y = 1$

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
- $= \frac{\cos t + 4\cos(2t)}{-\sin t - 4\sin(2t)}$.
- When $t = \pi/2$, we have $\frac{dy}{dx} = \frac{-4}{-1} = 4$.
- Also, when $t = \pi/2$, the corresponding point on the curve is $(-2, 1)$.
- Therefore, when $t = \pi/2$, the tangent line has equation $y - 1 = 4(x + 2)$ or $y = 4x + 9$.

Question 22

22. Which integral below gives the arclength of the curve $x = 1 - 2\cos t$, $y = \sin^2\left(t/2\right)$, $0 \le t \le \pi$?

a) $\int_0^\pi \sqrt{4\sin^2 t + \sin^2(t/2)\cos^2(t/2)}\ dt$

b) $\int_0^\pi \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^4(t/2)}\ dt$

c) $\displaystyle\int_0^\pi \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t/2)\cos^2(t/2)}\ dt$

d) $\int_0^\pi \sqrt{4\sin^2 t + \sin^4(t/2)}\ dt$

e) $\int_0^\pi \sqrt{\sin^2(t/2) - 2\sin^2(t/2)\cos(t)}\ dt$

- $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$
- $x'(t) = 2\sin t$ and $y'(t) = \frac{2}{2}\sin(t/2)\cos(t/2)$.
- $L = \int_0^\pi \sqrt{4\sin^2 t + \sin^2(t/2)\cos^2(t/2)} dt$

Question 23

23. The point $(2, \frac{11\pi}{3})$ in polar coordinates corresponds to which point below in Cartesian coordinates?
$(1, -\sqrt{3}\,)$
$(-\sqrt{3}, 1)$
$(-1, \sqrt{3}\,)$
$(\sqrt{3}, -1)$
Since $\dfrac{11\pi}{3} > 2\pi$, there is no such point.

- $x = r\cos\theta = 2\cos(11\pi/3) = 2\cos(5\pi/3) = 1$
- $y = r\sin\theta = 2\sin(11\pi/3) = 2\sin(11\pi/3) = 2(-\sqrt{3}/2) = -\sqrt{3}$
- Therefore the point in Cartesian coordinates is $(1, -\sqrt{3})$.

Question 24

24. Find the equation for the tangent line to the curve with polar equation: $r = 2 - 2\cos\theta$ at the point $\theta = \pi/2$.

$y = 2 - x$

$y = 2 - \pi + 2x$

$y = 2 + \dfrac{\pi}{2} - x$

$y = 2 + 2x$

$y = 0$

- A parameterization of this curve is given by
  $x = r\cos\theta = (2 - 2\cos\theta)\cos\theta = 2\cos\theta - 2\cos^2\theta$.
  $y = r\sin\theta = (2 - 2\cos\theta)\sin\theta = 2\sin\theta - 2\cos\theta\sin\theta$
- The slope at any point on the curve is given by
  $\frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta - 2[-\sin^2\theta + \cos^2\theta]}{-2\sin\theta - 4\cos\theta\sin\theta} = \frac{2\cos\theta + 2\sin^2\theta - 2\cos^2\theta}{-2\sin\theta + 4\sin\theta\cos\theta}$.
- When $\theta = \pi/2$, we get $\frac{dy/d\theta}{dx/d\theta} = \frac{0+2-0}{-2} = -1$.
- When $\theta = \pi/2$, the corresponding point on the curve is given by $x = 0$ and $y = 2$.
- Therefore the tangent is given by $y - 2 = -x$ or $y = 2 - x$.

Question 25

25. Find the length of the polar curve between $\theta = 0$ and $\theta = 2\pi$

$$r = e^{-\theta}.$$

$\sqrt{2}(1 - e^{-2\pi})$
$\frac{1}{4}(1 - e^{-4\pi})$
$2e^{-4\pi}$
$2 - e^{-2\pi}$
$2\pi(1 + e^{-2\pi})$

- The length of the polar curve is given by $L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$
- $\underset{=}{dr/d\theta} - e^{-\theta}, \quad \alpha = 0, \qquad \beta = 2\pi.$
- $L = \int_0^{2\pi} \sqrt{e^{-2\theta} + e^{-2\theta}} d\theta = \int_0^{2\pi} e^{-\theta} \sqrt{2} d\theta = \sqrt{2}[-e^{-\theta}]_0^{2\pi} =$
  $\sqrt{2}[-e^{-2\pi} + e^0] = \sqrt{2}[1 - e^{-2\pi}].$